

A Diophantine Connection to the Collatz Conjecture

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Abstract

If the Collatz conjecture is false, it could be falsified in one of two ways. One is to show that for some starting positive integer, the trajectory is unbounded; the other is to show that the trajectory reaches a loop other than the familiar $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$. In this article we show that the latter assertion is equivalent to a finding a simultaneous solution to a countable collection of Diophantine multinomials.

1 Background

The Collatz Conjecture is named for the mathematician Lothar Collatz, who introduced it in 1937.[1] J. Lagarias provided a useful survey of the subject.[2] P. Erdős remarked that “Mathematics may not be ready for such problems.”[3] By 2020, the conjecture had been verified by computer for all starting values up to 2^{68} . [4]

2 Introduction

The principal result of this article is that every Collatz loop $s_1 \rightarrow s_2 \rightarrow \cdots \rightarrow s_N \rightarrow s_1$ corresponds to a countably infinite system of Diophantine equations in the $2N$ variables $\{t_1, t_2, \cdots, t_N, u_1, u_2, \cdots, u_N\}$. This means there exists an assignment of positive integers to the variables $\{t_1, t_2, \cdots, t_N, u_1, u_2, \cdots, u_N\}$ such that **every** one of the countably infinitely many Diophantine equations

in the corresponding system is satisfied by **that** assignment of positive integers.

At this writing only one Collatz loop is known, namely $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$. Proving that no others exist might be accomplished by showing that only **this** Collatz loop satisfies **all** of the countably infinitely many corresponding Diophantine equations.

Definition 1. The Collatz function C is given by

$$C(n) \equiv \begin{cases} 3n + 1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

A Collatz loop is usually represented as a finite sequence in which the ordering reflects the action of the function C , i.e. $(4, 2, 1)$ or $4 \rightarrow 2 \rightarrow 1$. However, there is a bijection between the Collatz loops and the sets of members of Collatz loops. For instance, the membership set $\{1, 2, 4\}$ corresponds uniquely to the Collatz loop $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$. No information is lost by representing a Collatz loop as a set rather than as an ordered sequence. In the following, the set representation will be used.

Definition 2. We say that a nonempty finite set of integers $S \subset \mathbb{Z}^+$ is a **Collatz set** if for some $s \in S$, $S = \{s, C(s), C^2(s), \dots, C^n(s)\}$, where n is the smallest positive integer such that $C^{n+1}(s) = s$. Such an s is called a **generator** of S . Note that since $C(j) \neq j$ for every $j \in \mathbb{Z}^+$, no Collatz set is a singleton set.

Note that $\{1, 2, 4\}$ is a Collatz set (the only one known at this writing).

3 Propositions

Proposition 1. S is a Collatz set if and only if for **every** $s \in S$,

$$S = \{s, C(s), C^2(s), \dots, C^{|S|-1}(s)\}, \text{ and } C^{|S|}(s) = s.$$

Proof. ($\Rightarrow$) Suppose S is a Collatz set with $S = \{\alpha, C(\alpha), C^2(\alpha), \dots, C^{|S|-1}(\alpha)\}$, and $C^{|S|}(\alpha) = \alpha$. S cannot be a singleton set, so let β be some other member of S . There exists some $j \in \{1, 2, \dots, |S| - 1\}$ such that $\beta = C^j(\alpha)$. Note

that

$$\begin{aligned}
S &= \{\alpha, C(\alpha), \dots, C^{j-1}(\alpha), C^j(\alpha), C^{j+1}(\alpha), \dots, C^{|S|-1}(\alpha)\} \\
&= \{C^j(\alpha), C^{j+1}(\alpha), \dots, C^{|S|-1}(\alpha), \alpha, \dots, C^{j-1}(\alpha)\} \\
&= \{C^j(\alpha), \dots, C^{|S|-1}(\alpha), C^{|S|}(\alpha), \dots, C^{j-1}(C^{|S|}(\alpha))\} \\
&= \{C^j(\alpha), \dots, C^{|S|-1}(\alpha), C^{|S|}(\alpha), \dots, C^{j+|S|-1}(\alpha)\} \\
&= \{C^j(\alpha), C^{j+1}(\alpha), \dots, C^{j+|S|-1}(\alpha)\} \\
&= \{C^j(\alpha), C(C^j(\alpha)), \dots, C^{|S|-1}(C^j(\alpha))\} \\
&= \{\beta, C(\beta), C^2(\beta), \dots, C^{|S|-1}(\beta)\}.
\end{aligned} \tag{1}$$

Also, $C^{|S|}(\beta) = C^{|S|}(C^j(\alpha)) = C^j(C^{|S|}(\alpha)) = C^j(\alpha) = \beta$.

($\Leftarrow$) Suppose for every $s \in S$, $S = \{s, C(s), C^2(s), \dots, C^{|S|-1}(s)\}$, and $C^{|S|}(s) = s$. Fix $\alpha \in S$. Then $S = \{\alpha, C(\alpha), C^2(\alpha), \dots, C^n(\alpha)\}$, and $C^{|S|}(\alpha) = \alpha$, and so S is a Collatz set. $\square$

This implies an equivalent alternative definition of Collatz set:

Definition 3. We say that a nonempty finite set of integers $S \subset \mathbb{Z}^+$ is a **Collatz set** if for **every** $s \in S$, $S = \{s, C(s), C^2(s), \dots, C^n(s)\}$, where n is the smallest positive integer such that $C^{n+1}(s) = s$. Such an s is called a **generator** of S .

Proposition 2. If $X \subset \mathbb{Z}^+$ is finite and nonempty, and $f : X \rightarrow X$ is an injection, then f is a permutation of X .

Proof. Since f is an injection, $|f(X)| = |X|$. Since $f(X) \subseteq X$, we must have $f(X) = X$, so f is a surjection, hence f is a permutation. $\square$

Proposition 3. If S is a Collatz set, then $C(S) = S$ and $C|_S$ is a permutation of S .

Proof. By definition of Collatz set, for some $\alpha \in S$ and some $n \in \mathbb{Z}^+$ we may write

$$S = \{\alpha, C(\alpha), \dots, C^n(\alpha)\}. \tag{2}$$

with $C^{n+1}(\alpha) = \alpha$. This implies

$$\begin{aligned}
C(S) &= \{C(\alpha), C^2(\alpha), \dots, C^n(\alpha), C^{n+1}(\alpha)\} \\
&= \{C(\alpha), C^2(\alpha), \dots, C^n(\alpha), \alpha\} \\
&= \{\alpha, C(\alpha), C^2(\alpha), \dots, C^n(\alpha)\} = S.
\end{aligned} \tag{3}$$

Next, suppose $s_1, s_2 \in S$ and $C|_S(s_1) = C|_S(s_2)$. By Proposition 1, for $n = |S| - 1$ we can write $S = \{s_1, C(s_1), C^2(s_1), \dots, C^n(s_1)\}$, and $C^{n+1}(s_1) = s_1$. We also may write $S = \{s_2, C(s_2), C^2(s_2), \dots, C^n(s_2)\}$, and $C^{n+1}(s_2) = s_2$. Since $C|_S(s_1) = C|_S(s_2)$, we have $C(s_1) = C(s_2)$. This implies $C^n(C(s_1)) = C^n(C(s_2))$, i.e. $C^{n+1}(s_1) = C^{n+1}(s_2)$. Thus $s_1 = s_2$, so $C|_S : S \rightarrow S$ is an injection. Since $C|_S(S) = C(S) = S$, $C|_S$ is an injection of the finite set S into itself. Thus by Proposition 2, $C|_S$ is a permutation of S . $\square$

Proposition 4. *Let finite nonempty set $X \subset \mathbb{Z}^+$.*

Let $g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$. Then $g|_X$ is a permutation of X if and only if

$$\sum_{s \in X} \frac{p(g(s))}{q(g(s))} \equiv \sum_{s \in X} \frac{p(s)}{q(s)} \quad (4)$$

for any $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$.

Proof. ($\Rightarrow$) Suppose $g|_X$ is a permutation of X . Fix $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$. Then

$$\begin{aligned} \sum_{s \in X} \frac{p(g(s))}{q(g(s))} &= \sum_{t \in g|_X(X)} \frac{p(g(g|_X^{-1}(t)))}{q(g(g|_X^{-1}(t)))} = \sum_{t \in g(X)} \frac{p(g(g|_X^{-1}(t)))}{q(g(g|_X^{-1}(t)))} \\ &= \sum_{t \in X} \frac{p(g(g|_X^{-1}(t)))}{q(g(g|_X^{-1}(t)))} = \sum_{t \in X} \frac{p(g|_X(g|_X^{-1}(t)))}{q(g|_X(g|_X^{-1}(t)))} = \sum_{t \in X} \frac{p(t)}{q(t)}. \end{aligned} \quad (5)$$

($\Leftarrow$) Suppose

$$\sum_{s \in X} \frac{p(g(s))}{q(g(s))} \equiv \sum_{s \in X} \frac{p(s)}{q(s)} \quad (6)$$

for any $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$. Let $T \equiv X \cup g(X)$. Let $a = \min(T)$ and $b = \max(T)$.

Fix $c \in X$. Since $X \subseteq T \subseteq \{a, a+1, \dots, b\}$, we have $c \in \{a, a+1, \dots, b\}$.

Let $p(x) \equiv \prod_{\substack{m=a \\ m \neq c}}^b (x-m)$ and $q(x) \equiv \prod_{\substack{m=a \\ m \neq c}}^b (c-m)$. Then $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$

and $0 \notin q(X)$.

For any $s \in X$, we have

$$\frac{p(s)}{q(s)} = \begin{cases} 1 & \text{if } s = c, \\ 0 & \text{if } s \in X \setminus \{c\}. \end{cases} \quad (7)$$

Thus

$$\sum_{s \in X} \frac{p(s)}{q(s)} = \sum_{\substack{s \in X \\ s=c}} |\{s \in X; s=c\}| = 1. \quad (8)$$

For any $s \in X$, we have $g(s) \in T$, so

$$\frac{p(g(s))}{q(g(s))} = \begin{cases} 1 & \text{if } g(s) = c, \\ 0 & \text{if } g(s) \in T \setminus \{c\}. \end{cases} \quad (9)$$

Thus

$$\sum_{s \in X} \frac{p(g(s))}{q(g(s))} = \sum_{\substack{s \in X \\ g(s)=c}} 1 = |\{s \in X; g(s) = c\}|. \quad (10)$$

Finally, the hypothesis implies

$$|\{s \in X; g(s) = c\}| = \sum_{s \in X} \frac{p(g(s))}{q(g(s))} = \sum_{s \in X} \frac{p(s)}{q(s)} = 1, \quad (11)$$

so we get

$$|\{s \in X; g(s) = c\}| = 1. \quad (12)$$

Therefore there must exist exactly one member $s \in X$ such that $g(s) = c$. Since our choice of $c \in X$ was arbitrary, every member of X is expressible as the image under g of exactly one member of X . Hence there exists a unique mapping $h : X \rightarrow X$ such that $g(h(s)) = s$ for every $s \in X$. Next, suppose $h(s_1) = h(s_2)$ for some $s_1, s_2 \in X$. Then $s_1 = g(h(s_1)) = g(h(s_2)) = s_2$, so h must be an injection of the finite set X into itself, therefore by Proposition 2, h is a permutation of X . Thus $h^{-1} : X \rightarrow X$ exists and is a permutation of X . Finally, for any $s \in X$,

$h^{-1}(s) = g(h(h^{-1}(s))) = g(s) = g|_X(s)$, so in fact $g|_X \equiv h^{-1}$, thus $g|_X$ is a permutation of X . $\square$

Proposition 5. *Distinct Collatz sets must be disjoint.*

Proof. Suppose distinct Collatz sets S_1 and S_2 are not disjoint. Fix $\alpha \in S_1 \cap S_2$. Then $\alpha \in S_1$, so by Proposition 1, for some $n \in \mathbb{Z}^+$ we have $S_1 = \{\alpha, C(\alpha), C^2(\alpha), \dots, C^n(\alpha)\}$, and $C^{n+1}(\alpha) = \alpha$. Next, $\alpha \in S_2$, so for the same $n \in \mathbb{Z}^+$, we have $S_2 = \{\alpha, C(\alpha), C^2(\alpha), \dots, C^n(\alpha)\}$. This implies $S_1 = S_2$, contradicting our hypothesis that S_1 and S_2 are distinct. Thus S_1 and S_2 are disjoint. $\square$

Proposition 6. *Let finite nonempty set $X \subset \mathbb{Z}^+$. $C|_X$ is a permutation of X if and only if X is a disjoint union of Collatz sets.*

Proof. ($\Rightarrow$) Suppose $C|_X$ is a permutation of X . Let $L : X \rightarrow \mathcal{P}(X)$ be given by $L(s) \equiv \{s, C(s), C^2(s), \dots\}$. Note that $C(X) \subseteq X$. Indeed, a straightforward induction argument verifies $C^k(X) \subseteq X$ for every $k \in \mathbb{Z}^+$. This implies $L(s) \subseteq X$ for every $s \in X$. Fix $\alpha \in X$. Since $L(\alpha) \subseteq X$, $L(\alpha)$ is finite. Since $L(\alpha)$ is finite, we may write $L(\alpha) \equiv \{\alpha, C(\alpha), C^2(\alpha), \dots, C^{|L(\alpha)|-1}(\alpha)\}$ with $C^{|L(\alpha)|}(\alpha) = \alpha$. Thus $L(\alpha)$ is a Collatz set. Since $\alpha \in X$ was arbitrary, we may conclude that $L(s)$ is a Collatz set for every $s \in X$. Since $s \in L(s)$ for every $s \in X$, we have $X \subset \bigcup_{s \in X} L(s)$. Clearly $\bigcup_{s \in X} L(s) \subset X$, since each $L(s) \subseteq X$, thus we get $X = \bigcup_{s \in X} L(s)$, a cover of X .

Finally, we may find some $X' \subseteq X$ of minimal cardinality such that $S = \bigcup_{s \in X'} L(s)$. Its minimality ensures that the members of the subcover are distinct. This is a disjoint subcover since by Proposition 5 distinct Collatz sets must be disjoint. Therefore X is a disjoint union of Collatz sets.

($\Leftarrow$) Suppose X is a disjoint union of Collatz sets. We may write

$$X = \bigcup_{j=1}^J X_j$$
 with disjoint Collatz sets $X_j \subseteq X$. By Proposition 3 we have $C(X_j) = X_j$ for each j , and each $C|_{X_j}$ is a permutation of X_j . Thus
$$C(X) = C\left(\bigcup_{j=1}^J X_j\right) = \bigcup_{j=1}^J C(X_j).$$
 Each $C(X_j) = X_j$, hence
$$\bigcup_{j=1}^J C(X_j) = \bigcup_{j=1}^J X_j = S,$$
 so we get $C(X) = X$. Finally, we note that $C|_X(s) = C|_{X_j}(s)$ where $s \in X_j$.

Next, we note that $C|_X$ must be an injection. Indeed, suppose

$C|_X(s_1) = C|_X(s_2)$. Because $\{X_1, X_2, \dots, X_J\}$ is a disjoint cover of X and $C(X_j) = X_j$ for all j , we must have $s_1, s_2 \in X_k$ for some unique k , and also $C|_{X_k}(s_1) = C|_{X_k}(s_2)$. Since $C|_{X_k}$ is an injection on X_k , we get that $s_1 = s_2$, so $C|_X$ is an injection. Therefore by Proposition 2, $C|_X$ is a permutation of X . $\square$

Proposition 7. *Let X be a nonempty finite subset of $\mathbb{Z}^+$. Then X is a disjoint union of Collatz sets if and only if for every $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$*

with $0 \notin q(X)$ we have

$$\sum_{s \in X} \left(\begin{aligned} &(1 - (s - 2 \lfloor \frac{s}{2} \rfloor)) \frac{p(\lfloor \frac{s}{2} \rfloor)}{q(\lfloor \frac{s}{2} \rfloor)} \\ &- \frac{p(s)}{q(s)} + (s - 2 \lfloor \frac{s}{2} \rfloor) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) = 0. \quad (13)$$

Proof. ($\Rightarrow$) Suppose $X \equiv \bigcup_{k=1}^K X_k$ is a disjoint union of Collatz sets X_k , and fix $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$.

Because each X_k is itself a disjoint union of Collatz sets, namely a union consisting of X_k alone, by Proposition 6, $C|_{X_k}$ is a permutation of X_k . Thus by Proposition 4, for each $k \in \{1, 2, \dots, K\}$, we must have

$$\begin{aligned} \sum_{s \in X_k} \frac{p(s)}{q(s)} &= \sum_{s \in X_k} \frac{p(C(s))}{q(C(s))} = \sum_{\substack{s \in X_k \\ s \text{ even}}} \frac{p(C(s))}{q(C(s))} + \sum_{\substack{s \in X_k \\ s \text{ odd}}} \frac{p(C(s))}{q(C(s))} \\ &= \sum_{\substack{s \in X_k \\ s \text{ even}}} \frac{p(s/2)}{q(s/2)} + \sum_{\substack{s \in X_k \\ s \text{ odd}}} \frac{p(3s+1)}{q(3s+1)} \\ &= \sum_{s \in X_k} \left(\frac{1}{2}(1 + (-1)^s) \frac{p(s/2)}{q(s/2)} + \frac{1}{2}(1 - (-1)^s) \frac{p(3s+1)}{q(3s+1)} \right), \end{aligned} \quad (14)$$

whence

$$\sum_{s \in X_k} \left(\begin{aligned} &\frac{1}{2}(1 + (-1)^s) \frac{p(s/2)}{q(s/2)} \\ &- \frac{p(s)}{q(s)} + \frac{1}{2}(1 - (-1)^s) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) = 0. \quad (15)$$

Now note that for any $s \in \mathbb{Z}$, $(-1)^s \equiv (-1)^{s-2\lfloor s/2 \rfloor} \equiv 1 - 2(s - 2\lfloor s/2 \rfloor)$, so $\frac{1}{2}(1 + (-1)^s) \equiv 1 - (s - 2\lfloor s/2 \rfloor)$ and $\frac{1}{2}(1 - (-1)^s) \equiv s - 2\lfloor s/2 \rfloor$. Thus,

$$\sum_{s \in X_k} \left(\begin{aligned} &(1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} \\ &- \frac{p(s)}{q(s)} + (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) = 0. \quad (16)$$

Since X is the disjoint union of disjoint Collatz sets X_k , we have

$$\begin{aligned} \sum_{s \in X} \left(\begin{aligned} &(1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} \\ &- \frac{p(s)}{q(s)} + (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) \\ &= \sum_{k=1}^K \sum_{s \in X_k} \left(\begin{aligned} &(1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} \\ &- \frac{p(s)}{q(s)} + (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) = \sum_{k=1}^K 0 = 0. \end{aligned} \quad (17)$$

($\Leftarrow$) Suppose for every $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$ we have

$$\sum_{s \in X} \left((1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} - \frac{p(s)}{q(s)} + (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \right) = 0. \quad (18)$$

Next,

$$\begin{aligned} \sum_{s \in X} \frac{p(C(s))}{q(C(s))} - \sum_{s \in X} \frac{p(s)}{q(s)} &= \sum_{\substack{s \in X \\ s \text{ even}}} \frac{p(C(s))}{q(C(s))} + \sum_{\substack{s \in X \\ s \text{ odd}}} \frac{p(C(s))}{q(C(s))} - \sum_{s \in X} \frac{p(s)}{q(s)} \\ &= \sum_{\substack{s \in X \\ s \text{ even}}} \frac{p(C(s))}{q(C(s))} - \sum_{s \in X} \frac{p(s)}{q(s)} + \sum_{\substack{s \in X \\ s \text{ odd}}} \frac{p(C(s))}{q(C(s))} \\ &= \sum_{\substack{s \in X \\ s \text{ even}}} \frac{p(s/2)}{q(s/2)} - \sum_{s \in X} \frac{p(s)}{q(s)} + \sum_{\substack{s \in X \\ s \text{ odd}}} \frac{p(3s+1)}{q(3s+1)} \\ &= \sum_{s \in X} (1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} - \sum_{s \in X} \frac{p(s)}{q(s)} \\ &\quad + \sum_{s \in X} (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} = 0, \end{aligned} \quad (19)$$

and thus

$$\sum_{s \in X} \frac{p(C(s))}{q(C(s))} = \sum_{s \in X} \frac{p(s)}{q(s)}. \quad (20)$$

By Proposition 4, we get that $C|_X$ is a permutation of X . Finally, by Proposition 6, X is a disjoint union of Collatz sets. $\square$

4 Diophantine Equations

Proposition 8. *Let X be a nonempty finite subset of $\mathbb{Z}^+$. Then X is a disjoint union of Collatz sets if and only if there exists some*

$\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $p \in \mathbb{Z}[x]$ and

$q \in \mathbb{Z}[x]$ with $0 \notin q(X)$ we have

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{s_j}{2}\right)}{q\left(\frac{s_j}{2}\right)} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - s_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0. \quad (21)$$

Proof. ($\Rightarrow$) By Proposition 7, we have

$$\sum_{s \in X} \left(\begin{aligned} &(1 - (s - 2\lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} \\ &- \frac{p(s)}{q(s)} + (s - 2\lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \end{aligned} \right) = 0. \quad (22)$$

Now write $X = \{s_1, s_2, \dots, s_{|X|}\}$. The indexing of elements of X may be chosen arbitrarily. We may now take $t_j \equiv \lfloor s_j/2 \rfloor$ and $u_j = s_j - 2\lfloor s_j/2 \rfloor$. Substituting in Equation 22, we get

$$\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{s_j}{2}\right)}{q\left(\frac{s_j}{2}\right)} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) = 0, \quad (23)$$

and by definition of the t_j and u_j we have

$$\sum_{j=1}^{|X|} ((2t_j + u_j - s_j)^2 + (1 - u_j)^2 u_j^2) = 0. \quad (24)$$

Equation 23 implies

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{s_j}{2}\right)}{q\left(\frac{s_j}{2}\right)} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) \right)^2 = 0. \quad (25)$$

Equations 24 and 25 together yield

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{s_j}{2}\right)}{q\left(\frac{s_j}{2}\right)} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - s_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0, \quad (26)$$

as desired.

($\Leftarrow$) Suppose there exists some

$\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$, we have

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p(\frac{s_j}{2})}{q(\frac{s_j}{2})} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - s_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0. \quad (27)$$

Equation 27 implies

$$\sum_{j=1}^{|X|} ((2t_j + u_j - s_j)^2 + (1 - u_j)^2 u_j^2) = 0, \quad (28)$$

which implies $s_j \equiv 2t_j + u_j$ and $u_j \in \{0, 1\}$ for $j \in \{1, 2, \dots, |X|\}$. These conditions yield $t_j \equiv \lfloor \frac{s_j}{2} \rfloor$ and $u_j \equiv s_j - 2 \lfloor \frac{s_j}{2} \rfloor$ for $j \in \{1, 2, \dots, |X|\}$. Equation 27 also implies

$$\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p(\frac{s_j}{2})}{q(\frac{s_j}{2})} - \frac{p(s_j)}{q(s_j)} + u_j \frac{p(3s_j + 1)}{q(3s_j + 1)} \right) = 0, \quad (29)$$

and the definitions of the t_j and u_j yield

$$\sum_{s \in X} \left((1 - (s - 2 \lfloor s/2 \rfloor)) \frac{p(s/2)}{q(s/2)} - \frac{p(s)}{q(s)} + (s - 2 \lfloor s/2 \rfloor) \frac{p(3s+1)}{q(3s+1)} \right) = 0. \quad (30)$$

We now invoke Proposition 7 to conclude that X is a disjoint union of Collatz sets, as desired. $\square$

Proposition 9. *If $X \equiv \{x_1, x_2, \dots, x_{|X|}\}$ is a Collatz set, then there exists some $\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $p \in \mathbb{Z}[x]$ and*

$q \in \mathbb{Z}[x]$ with $0 \notin q(X)$, we have

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{x_j}{2}\right)}{q\left(\frac{x_j}{2}\right)} - \frac{p(x_j)}{q(x_j)} + u_j \frac{p(3x_j + 1)}{q(3x_j + 1)} \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0. \quad (31)$$

Proof. Suppose $X \equiv \{x_1, x_2, \dots, x_{|X|}\}$ is a Collatz set. Then X is the disjoint union of Collatz sets, namely the Collatz set X alone. Therefore, by Proposition 8, there exists some $\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$, we have

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{x_j}{2}\right)}{q\left(\frac{x_j}{2}\right)} - \frac{p(x_j)}{q(x_j)} + u_j \frac{p(3x_j + 1)}{q(3x_j + 1)} \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0, \quad (32)$$

as desired. $\square$

The essence of Proposition 9 is that to each Collatz set $\{x_1, x_2, \dots, x_{|X|}\}$ there corresponds a countably infinite system of Diophantine equations which must have a common solution. Each equation corresponds to some pair of polynomials p and q in $\mathbb{Z}[x]$.

In particular, if we take $q \equiv 1$, and $p \in \mathbb{Z}[x]$ with $p \not\equiv 0$, every equation

$$\left(\sum_{j=1}^{|X|} \left((1 - u_j) p\left(\frac{x_j}{2}\right) - p(x_j) + u_j p(3x_j + 1) \right) \right)^2 + \sum_{j=1}^{|X|} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0 \quad (33)$$

must have a solution.

A practical application of this result might be to find a proof that the familiar $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$ is the only Collatz loop for which all of the corresponding Diophantine equations have a common solution.

5 Special Case

Proposition 10. *If $X = \{x_1, x_2, \dots, x_{|X|}\}$ is a Collatz set, then there exists some $\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $m \in \mathbb{Z}^+$ we have*

$$\begin{aligned} & \sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 \\ & + \left(\sum_{j=1}^{|X|} (u_j ((6x_j + 2)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 \\ & + \sum_{j=1}^{|X|} 2^{2m} (1 - u_j)^2 u_j^2 = 0. \end{aligned} \quad (34)$$

Proof. Suppose $X = \{x_1, x_2, \dots, x_{|X|}\}$ is a Collatz set. The indexing of elements of X may be chosen arbitrarily. By Proposition 9 there exists some $\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $p \in \mathbb{Z}[x]$ and $q \in \mathbb{Z}[x]$ with $0 \notin q(X)$, we have

$$\begin{aligned} & \left(\sum_{j=1}^{|X|} \left((1 - u_j) \frac{p\left(\frac{x_j}{2}\right)}{q\left(\frac{x_j}{2}\right)} - \frac{p(x_j)}{q(x_j)} + u_j \frac{p(3x_j + 1)}{q(3x_j + 1)} \right) \right)^2 \\ & + \sum_{j=1}^{|X|} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0. \end{aligned} \quad (35)$$

Fix $m \in \mathbb{Z}^+$. If we take $p(t) \equiv t^m$ and $q(t) \equiv 1$, we get

$$\begin{aligned} & \left(\sum_{j=1}^{|X|} u_j ((3x_j + 1)^m - 2^{-m} x_j^m) - (1 - 2^{-m}) x_j^m \right)^2 \\ & + \sum_{j=1}^{|X|} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0. \end{aligned} \quad (36)$$

To get an expression with integer coefficients, we multiply through by 2^{2m} ,

obtaining

$$\left(\sum_{j=1}^{|X|} (u_j (2^m(3x_j + 1)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 \\ \sum_{j=1}^{|X|} 2^{2m} (2t_j + u_j - x_j)^2 + \sum_{j=1}^{|X|} 2^{2m} (1 - u_j)^2 u_j^2 = 0, \quad (37)$$

as desired. $\square$

Lemma 1. Suppose $\{u_1, u_2, \dots, u_L\} \subset \mathbb{R}$. If

$$\sum_{j=1}^L (1 - u_j)^2 u_j^2 = 0, \quad (38)$$

then for any $a, b \in \mathbb{R}$ and $c \in \mathbb{Z}^+$, we have

$$(a + bu_j)^c \equiv a^c(1 - u_j) + (a + b)^c u_j. \quad (39)$$

for every $j \in \{1, 2, \dots, L\}$.

Proof. The hypothesis implies for every $j \in \{1, 2, \dots, L\}$, $(1 - u_j)^2 u_j^2 = 0$ and therefore either $u_j = 0$ or $u_j = 1$, so we must have $u_j^2 = u_j$ for every $j \in \{1, 2, \dots, L\}$. Next, we proceed with induction on c . First note that $(a + bu_j)^1 = a + bu_j = a^1(1 - u_j) + (a + b)^1 u_j$, so the case $c = 1$ is verified. Next, suppose $(a + bu_j)^c \equiv a^c(1 - u_j) + (a + b)^c u_j$ for some $c \in \mathbb{Z}^+$. Then

$$\begin{aligned} (a + bu_j)^{c+1} &= (a + bu_j)(a + bu_j)^c = (a + bu_j)(a^c(1 - u_j) + (a + b)^c u_j) \\ &= (a + bu_j)a^c(1 - u_j) + (a + bu_j)(a + b)^c u_j \\ &= a^c((a + bu_j) - (a + bu_j)u_j) + (au_j + bu_j^2)(a + b)^c \\ &= a^c(a + bu_j) - a^c(au_j + bu_j^2) + (au_j + bu_j^2)(a + b)^c. \end{aligned} \quad (40)$$

Since $u_j^2 = u_j$, we get

$$\begin{aligned} (a + bu_j)^{c+1} &= a^c(a + bu_j) - a^c(au_j + bu_j) + (au_j + bu_j)(a + b)^c \\ &= a^c(a + bu_j) - a^c(a + b)u_j + (a + b)(a + b)^c u_j \\ &= a^c(a + bu_j) - a^c(a + b)u_j + (a + b)^{c+1} u_j \\ &= a^{c+1} + a^c bu_j - a^{c+1} u_j - a^c bu_j + (a + b)^{c+1} u_j \\ &= a^{c+1} - a^{c+1} u_j + (a + b)^{c+1} u_j = a^{c+1} (1 - u_j) + (a + b)^{c+1} u_j, \end{aligned} \quad (41)$$

completing the induction. $\square$

Proposition 11. Fix $m \in \mathbb{Z}^+$, and some $X = \{x_1, x_2, \dots, x_{|X|}\}$. If

$$\begin{aligned} & \sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 \\ & + \left(\sum_{j=1}^{|X|} (u_j ((6x_j + 2)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 \\ & + \sum_{j=1}^{|X|} 2^{2m} (1 - u_j)^2 u_j^2 = 0, \quad (42) \end{aligned}$$

then

$$\sum_{j=1}^{|X|} (u_j ((2^m - 1)t_j^m - (2t_j + 1)^m + (6t_j + 4)^m) - (2^m - 1)t_j^m) = 0. \quad (43)$$

Proof. Suppose

$$\begin{aligned} & \sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 \\ & + \left(\sum_{j=1}^{|X|} (u_j ((6x_j + 2)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 \\ & + \sum_{j=1}^{|X|} 2^{2m} (1 - u_j)^2 u_j^2 = 0. \quad (44) \end{aligned}$$

Clearly $\sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0$, so Equation 44 reduces to

$$\begin{aligned} & \sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 \\ & + \left(\sum_{j=1}^{|X|} (u_j ((6x_j + 2)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 = 0, \quad (45) \end{aligned}$$

and $\sum_{j=1}^{|X|} (1 - u_j)^2 u_j^2 = 0$ with Lemma 1 implies

$$(2t_j + u_j)^m \equiv 2^m t_j^m (1 - u_j) + (1 + 2t_j)^m u_j. \quad (46)$$

Equation 45 implies $\sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 = 0$, so we get

$-x_j + 2t_j + u_j = 0$, i.e. $x_j = 2t_j + u_j$. Using this and Equation 46 with Equation 45 yields

$$\sum_{j=1}^{|X|} \left(\begin{array}{c} -2^m t_j^m (1 - u_j) - (1 + 2t_j)^m u_j \\ + 2^{-m} (1 - u_j) \left(\begin{array}{c} 2^m t_j^m (1 - u_j) \\ + (1 + 2t_j)^m u_j \end{array} \right) + u_j (1 + 3(2t_j + u_j))^m \end{array} \right) = 0. \quad (47)$$

Since $(1 + 3(2t_j + u_j)) \equiv 1 + 6t_j + 3u_j$, this becomes

$$\sum_{j=1}^{|X|} \left(\begin{array}{c} -2^m t_j^m (1 - u_j) - (1 + 2t_j)^m u_j + u_j (1 + 6t_j + 3u_j)^m \\ + 2^{-m} (1 - u_j) (2^m t_j^m (1 - u_j) + (1 + 2t_j)^m u_j) \end{array} \right) = 0. \quad (48)$$

Substitution into Equation 48 using the identity (via Lemma 1)

$$(1 + 6t_j + 3u_j)^m \equiv (1 + 6t_j)^m (1 - u_j) + (4 + 6t_j)^m u_j \quad (49)$$

and the equations

$$(1 - u_j)^2 \equiv 1 - u_j \quad (50)$$

and

$$(1 - u_j) u_j \equiv 0 \quad (51)$$

yields

$$\sum_{j=1}^{|X|} \left(\begin{array}{c} -2^m t_j^m (1 - u_j) - (1 + 2t_j)^m u_j + u_j \left(\begin{array}{c} (1 + 6t_j)^m (1 - u_j) \\ + (4 + 6t_j)^m u_j \end{array} \right) \\ + 2^{-m} (1 - u_j) (2^m t_j^m (1 - u_j) + (1 + 2t_j)^m u_j) \end{array} \right) = 0. \quad (52)$$

Expanding and using the equation $u_j^2 = u_j$, we obtain

$$\sum_{j=1}^{|X|} \left(\begin{array}{c} t_j^m - 2^m t_j^m + \left(\begin{array}{c} -2t_j^m + 2^m t_j^m - (1 + 2t_j)^m \\ + 2^{-m} (1 + 2t_j)^m + (1 + 6t_j)^m \end{array} \right) u_j + \\ (t_j^m - 2^{-m} (1 + 2t_j)^m - (1 + 6t_j)^m + (4 + 6t_j)^m) u_j \end{array} \right) = 0. \quad (53)$$

Collecting on u_j gives

$$\sum_{j=1}^{|X|} (u_j ((2^m - 1)t_j^m - (2t_j + 1)^m + (6t_j + 4)^m) - (2^m - 1)t_j^m) = 0, \quad (54)$$

as desired. $\square$

Proposition 12. *If $X = \{x_1, x_2, \dots, x_{|X|}\}$ is a Collatz set, then for $\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ with $x_j = 2t_j + u_j$ and $u_j \in \{0, 1\}$ and every $m \in \mathbb{Z}^+$ we have*

$$\sum_{j=1}^{|X|} (u_j ((2^m - 1)t_j^m - (2t_j + 1)^m + (6t_j + 4)^m) - (2^m - 1)t_j^m) = 0. \quad (55)$$

Proof. By Proposition 10, we have that there exists some

$\{t_1, t_2, \dots, t_{|X|}, u_1, u_2, \dots, u_{|X|}\} \subset \mathbb{Z}^+$ such that for every $m \in \mathbb{Z}^+$ we have

$$\begin{aligned} \sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 \\ + \left(\sum_{j=1}^{|X|} (u_j (2^m (3x_j + 1)^m - x_j^m) - (2^m - 1) x_j^m) \right)^2 \\ + \sum_{j=1}^{|X|} 2^{2m} (1 - u_j)^2 u_j^2 = 0. \end{aligned} \quad (56)$$

Clearly this condition implies $\sum_{j=1}^{|X|} 2^{2m} (-x_j + 2t_j + u_j)^2 = 0$, hence we have the equation $x_j = 2t_j + u_j$. By Proposition 11, we get

$$\sum_{j=1}^{|X|} (u_j ((2^m - 1)t_j^m - (2t_j + 1)^m + (6t_j + 4)^m) - (2^m - 1)t_j^m) = 0, \quad (57)$$

as desired. $\square$

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